ON THE CONTROLLABILITY OF QUASI-LINEAR SYSTEMS

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Picard's classical method of consecutive approximations is used to solve simultaneously the problems on existence and determination of controls, under some constraints, which transfer a quasi-linear system from a given position in the phase space into the origin of that space within a fixed time. In addition we find the radius of convergence, radius of the sphere of controllability and the upper bound for those values of the parameter of nonlinear elements, for which the system under consideration is controllable (in the Kalman [1] sense). We consider one particular case and compute the quantities mentioned above, for a quasi-linear second order system.

1. Let the motion of some dynamic system be described by

$$\frac{dx_\nu}{dt} = \sum_{\mu=1}^{n} a_{\nu\mu}(t)\,x_\mu + \varepsilon\Psi_\nu(x_1,\ldots,x_n,\ t) + u_\nu(t), \qquad x_\nu(0) = z_\nu^{\,0}\ (\nu = 1,\ldots,n) \tag{1.1}$$

Here x_ν are the phase coordinates of the system under consideration: $a_{\nu\mu}(t)$ are the parameters of the system which are known and continuous functions of time: $\Psi_\nu(x_1,\ldots$ $\ldots, x_n,\ t)$ are nonlinear functions; $u_\rho(t)$ are the controls, the behavior of which is to be determined, and ϵ is a certain positive parameter.

Set of scalar equations (1.1) is equivalent to the matrix equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}(t)\,\mathbf{x} + \varepsilon\boldsymbol{\Psi}(\mathbf{x},\ t) + \mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{z}^0 \tag{1.2}$$

where $\mathbf{x}$, $\mathbf{A}(t)$, $\boldsymbol{\Psi}(\mathbf{x},\ t)$, $\mathbf{u}(t)$ and $\mathbf{z}^0$ are the following matrices:

$$\mathbf{x} = \|x_\nu\|(n\times 1), \quad \mathbf{A}(t) = \|a_{\nu\mu}(t)\|(n\times n), \quad \boldsymbol{\Psi}(x,\ t) = \|\Psi_\nu(x_1,\ldots,x_n,\ t)\|(n\times 1),$$

$$\mathbf{u}(t) = \|u_\nu(t)\|(n\times 1), \quad \mathbf{z}^0 = \|z_\nu^{\,0}\|(n\times 1)$$

Denoting by $\mathbf{X}(t)$ the normalized fundamental matrix of (1.2) when $\mathbf{u}(t) \equiv 0$ and $\epsilon = 0$, we can replace the matrix differential equation (1.2) which is nonlinear, with the following nonlinear matrix integral equation

$$\mathbf{x}(t,\ \mathbf{u}) = \mathbf{X}(t)\left[\mathbf{z}^0 + \varepsilon\int_0^t \mathbf{X}^{-1}(\sigma)\,\boldsymbol{\Psi}(\mathbf{x}(\sigma,\ \mathbf{u}),\ \sigma)\,d\sigma + \int_0^t \mathbf{X}^{-1}(\sigma)\,\mathbf{u}(\sigma)\,d\sigma\right] \tag{1.3}$$

where $\mathbf{X}^{-1}(\sigma)$ denotes the inverse of $\mathbf{X}(\sigma)$.

Let us limit ourselves to the controls constant over the interval $0 \le t \le t_1$

$$\mathbf{u}(t) = \mathbf{u} = \text{const} \tag{1.4}$$

and let us consider the problem of determination of a constant control vector $\mathbf{u} = \|u_\nu\|\cdot$ $\cdot(n\times 1)$, constrained by the inequality

$$|\mathbf{u}| = \Big(\sum_{\nu=1} u_\nu^2\Big)^{1/2} \leqslant U^* \tag{1.5}$$

the vector being such that

$$\mathbf{x}(t_1, \mathbf{u}) = 0 \tag{1.6}$$

is satisfied.

Here t_1 is the instant of time and U^* is a positive number, both specified in advance.

We know that the problem of this type dealing with controlled motion of dynamic systems for the case when the constant control vector $\mathbf{u}$ is not subject to the constraint (1.5), was first stated and solved by Roitenberg [2].

It is easy to show, using (1.3), that the condition (1.6) will be fulfilled when the vector $\mathbf{u}$ is chosen as to satisfy the following relations

$$\mathbf{u} = -\mathbf{W}^{-1}(t_1)\,\mathbf{z}^\circ - \varepsilon \mathbf{W}^{-1}(t_1) \int_0^{t_1} \mathbf{X}^{-1}(t)\,\mathbf{\Psi}(\mathbf{x}(t, \mathbf{u}), t)\,dt \tag{1.7}$$

$$\mathbf{x}(t, \mathbf{u}) = \mathbf{X}(t)\left[\mathbf{z}^\circ + \mathbf{W}(t)\,\mathbf{u} + \varepsilon \int_0^t \mathbf{X}^{-1}(\sigma)\,\mathbf{\Psi}(\mathbf{x}(\sigma, \mathbf{u}), \sigma)\,d\sigma\right]\left(\mathbf{W}(t) = \int_0^t \mathbf{X}^{-1}(\sigma)\,d\sigma\right)$$
$$0 \leqslant t \leqslant t_1 \tag{1.8}$$

Substitution of (1.7) into the right-hand side of (1.8) results in the nonlinear matrix integral equation obtained by Roitenberg in [2]. Solving the latter and inserting this solution into the right-hand side of (1.7) we obtain numerical values for the required vector $\mathbf{u}$. We do not know, however, whether the inequality (1.5) holds, but we find that such conditions exist which, when imposed on the right-hand side of the matrix equation (1.2), produce the following result: a constant control vector $\mathbf{u}$ obtained from (1.7) and (1.8), fulfils the inequality (1.5). These conditions are:

1°. Vector function $\mathbf{\Psi}(x, t)$ must be continuous in all arguments over the closed region $\mathbf{D}_\Delta^{t_1}(x, t)$

$$\mathbf{D}_\Delta^{t_1}(x, t) = \{|\mathbf{x}| \leqslant \Delta = (1+\delta)\,h^+h^-U^*t_1,\ 0 \leqslant t \leqslant t_1\} \tag{1.9}$$

of the $(n+1)$-dimensional space. Here

$$\delta = \frac{1}{h^-W^-t_1}, \quad h^+ = \max|\mathbf{X}(t)|, \quad h^- = \max|\mathbf{X}^{-1}(t)|, \quad W^- = |\mathbf{W}^{-1}(t_1)|, \quad t \in [0, t_1] \tag{1.10}$$

In (1.9) and (1.10) $|\mathbf{Z}|$ denotes the norm of the matrix $\mathbf{Z}$. Here and the following we shall take the norm as the root of the sum of squares of the elements of the matrix.

2°. In the region $\mathbf{D}_\Delta^{t_1}(\mathbf{x}, t)$ vector function $\Psi(\mathbf{x}, t)$ satisfies the Lipshits' r-condition in $\mathbf{x}$. This means, that for any two points $(\mathbf{x}'', t)$ and $(\mathbf{x}', t)$ of the region $\mathbf{D}_\Delta^{t_1}(\mathbf{x}, t)$, the condition

$$|\mathbf{\Psi}|(\mathbf{x}', t) - \mathbf{\Psi}(\mathbf{x}'', t)| \leqslant r|\mathbf{x}'' - \mathbf{x}'|, \qquad r = \text{const} \tag{1.11}$$

holds.

3°. In addition,

$$|\mathbf{z}^0| \leqslant \varkappa_\varepsilon = \frac{U^*}{W^-} - \varepsilon h^- K t_1, \quad K = \sup, \quad |\mathbf{\Psi}(\mathbf{x}, t)|, \quad (\mathbf{x}, t) \in \mathbf{D}_\Delta^{t_1}(\mathbf{x}, t) \tag{1.12}$$

4°. Parameter ϵ is defined by

$$0 < \varepsilon < \varepsilon_0 = \min\left\{\frac{1}{(1 + h^-W^-t_1)\,h^+h^-rt_1}, \frac{U^*}{h^-W^-Kt_1}\right\} \tag{1.13}$$

We shall show, that, when the above assumptions hold, there exists a constant control vector $\mathbf{u}$ satisfying the condition (1. 5) together with relations (1. 7) and (1. 8), and we shall use Picard's method of consecutive approximations. Taking

$$\mathbf{u}^{\circ} = -\mathbf{W}^{-1}(t_1)\mathbf{z}^{\circ}, \quad \mathbf{x}(t, \mathbf{u}^{\circ}) = \mathbf{X}(t)[\mathbf{z}^{\circ} + \mathbf{W}(t)\mathbf{u}^{\circ}], \quad 0 \leqslant t \leqslant t_1 \tag{1.14}$$

as the zero approximation, we see that $|\mathbf{u}^{\circ}| \leqslant U^*$, $\mathbf{x}(t, u_{\circ}^{\circ}) \in \mathrm{D}_{\Delta}^{t_1}(\mathbf{x}, t)$ (by condition 3°) and from (1. 14) it follows that $\mathbf{x}(t_1, \mathbf{u}^{\circ}) = 0$.

Let us now assume that the k-th approximation is found and that it is such, that

$$|\mathbf{u}^k| \leqslant U^*, \quad \mathbf{x}(t, \mathbf{u}^k) \in \mathrm{D}_{\Delta}^{t_1}(\mathbf{x}, t), \quad \mathbf{x}(t_1, \mathbf{u}^k) = 0 \tag{1.15}$$

Then the $(k+1)$-th approximation is given by

$$\mathbf{u}^{k+1} = \mathbf{u}^{\circ} - \varepsilon\mathbf{W}^{-1}(t_1)\int_0^{t_1} \mathbf{X}^{-1}(t)\,\Psi(\mathbf{x}(t, \mathbf{u}^k), t)\,dt \tag{1.16}$$

$$\mathbf{x}(t, \mathbf{u}^{k+1}) = \mathbf{X}(t)\left[\mathbf{z}^{\circ} + \mathbf{W}(t)\mathbf{u}^{k+1} + \varepsilon\int_0^{t} \mathbf{X}^{-1}(\sigma)\,\Psi(\mathbf{x}(\sigma, \mathbf{u}^k), \sigma)\,d\sigma\right], \quad 0 \leqslant t \leqslant t_1 \tag{1.17}$$

By virtue of condition 3° we can easily show that

$$|\mathbf{u}^{k+1}| \leqslant U^*, \qquad \mathbf{x}(t, \mathbf{u}^{k+1}) \in \mathrm{D}_{\Delta}^{t_1}(\mathbf{x}, t)$$

while (1. 16) and (1. 17) imply that $\mathbf{x}(t_1\, \mathbf{u}^{k+1}) = 0$.

Let us now consider the question of convergence of (1. 16) and (1. 17). We shall show later that (1. 17) is equivalent to the problem of convergence of the series

$$\sum_{k=0}^{\infty} [\mathbf{x}(t, \mathbf{u}^{k+1}) - \mathbf{x}(t, \mathbf{u}^k)]$$

In its place, let us now consider the majorizing series

$$\sum_{k=0}^{\infty} \{\max_t |\mathbf{x}(t, \mathbf{u}^{k+1}) - \mathbf{x}(t, \mathbf{u}^k)|\} \qquad (t \in [0, t_1])$$

Using (1. 17) we can easily show that the following estimate holds:

$$\max_t |x(t, u^{k+1}) - \mathbf{x}(t, \mathbf{u}^k)| \leqslant \varepsilon h^+ h^- r t_1 \{\max_t |\mathbf{x}(t, \mathbf{u}^k) - \mathbf{x}(t, \mathbf{u}^{k-1})|\} + h^+ h^- t_1 |\mathbf{u}^{k+1} - \mathbf{u}^k| \quad (t \in [0, t_1]) \tag{1.18}$$

while (1. 16) yields

$$|\mathbf{u}^{k+1} - \mathbf{u}^k| \leqslant \varepsilon h^- W^- t_1 \{\max |\mathbf{x}(t, \mathbf{u}^k) - \mathbf{x}(t, \mathbf{u}^{k-1})|\} \quad (t \in [0, t_1]) \tag{1.19}$$

Last two relations give

$$\max_t |\mathbf{x}(t, \mathbf{u}^{k+1}) - \mathbf{x}(t, \mathbf{u}^k)| \leqslant \varepsilon (1 + h^- W^- t_1) h^+ h^- r t_1 \{\max_t |\mathbf{x}(t, \mathbf{u}^k) - \mathbf{x}(t, \mathbf{u}^{k-1})|\} \quad (t \in [0, t_1])$$

or, (by virtue of the choice of ε)

$$\frac{\max_t |\mathbf{x}(t, \mathbf{u}^{k+1}) - \mathbf{x}(t, \mathbf{u}^k)|}{\max_t |\mathbf{x}(t, \mathbf{u}^k) - \mathbf{x}(t, \mathbf{u}^{k-1})|} \leqslant \varepsilon(1 + h^- W^- t_1) h^+ h^- r t_1 < 1 \qquad (t \in [0, t_1]) \tag{1.20}$$

which shows that the majorizing series (by the D'Alembert criterion) converges. Hence, the sequence of approximations (1. 17) converges uniformly to some continuous vector function $\mathbf{x}(t, \mathbf{u}^*) \in \mathrm{D}_{\Delta}^{t_1}(\mathbf{x}, t)$, while the sequence of approximations (1. 16) converges by (1. 19), to some constant vector $\mathbf{u}^*$. At the same time $\mathbf{x}(t, \mathbf{u}^*)$ satisfies the integral matix equation (1. 8), while the constant vector $\mathbf{u}^*$ satisfies (1. 7).

Now we can easily see that $|\mathbf{u}^*| \leqslant U^*$ and $\mathbf{x}(t_1, \mathbf{u}^*) = 0$, which was to be proved.

2 . We shall now consider the case, when the number of controls is smaller than n. For definiteness let us deal with one control only. Let us assume that the motion of the system under consideration is described by the following matrix equation:

$$\frac{dx}{dt} = \mathbf{A}(t)\,\mathbf{x} + \varepsilon\Psi(\mathbf{x},\ t) + \mathbf{b}(t)\,u(t), \qquad \mathbf{x}(0) = z, \qquad \mathbf{b}(t) = \|b_\nu(t)\|\ (n \times 1) \quad (2.1)$$

Here $\mathbf{b}(t)$ is a known continuous vector function of time and $u(t)$ is the control representing a scalar function of time and constrained by the condition

$$|\,u(t)\,| \leqslant U^*, \quad 0 \leqslant t \leqslant t_1 \quad (2.2)$$

The matrix differential equation (2, 1) is equivalent to the following matrix integral equation

$$\mathbf{x}(t,\ u) = \mathbf{X}(t)\left[\mathbf{z}^\circ + \varepsilon \int_0^t \mathbf{X}^{-1}(\sigma)\,\Psi(\mathbf{x}(\sigma,\ u),\ \sigma)\,d\sigma + \int_0^t \mathbf{X}^{-1}(\sigma)\,\mathbf{b}(\sigma)\,u(\sigma)\,d\sigma\right] \quad (2.3)$$

We shall now solve the problem or control of the motion of the system (2, 1), i. e. the problem of establishing the law of variation with respect to time of the function $u(t)$ satisfying (2, 2) and transporting the system (2, 1) into the position

$$\mathbf{x}(t_1,\ u) = 0 \quad (2.4)$$

Following Roitenberg, we shall devide the time interval $0 \leq t \leq t_1$ into n equal or unequal subintervals and seek $u(t)$ in the form of a step function which is constant over each subinterval

$$\mu(t) = u_\nu = \text{const}, \qquad \sigma_{\nu-1} \leqslant t \leqslant \sigma_\nu \qquad (\nu = 1 \ldots,\ n,\ \sigma_0 = 0,\ \sigma_n = t_1) \quad (2.5)$$

Such a procedure results in an artificial increase of the number of controls until it is equal to the number of controlled coordinates. As a result, we obtain the following relations defining the vector $\mathbf{u} = \|\mathbf{u}_\nu\|\ (n \times 1)$,

$$\mathbf{u} = \mathbf{u}^\circ - \varepsilon \mathbf{W}_*^{-1}(t_1) \int_0^{t_1} \mathbf{X}^{-1}(t)\,\Psi(\mathbf{x}(t,\ \mathbf{u}),\ t)\,dt, \qquad \mathbf{u}^\circ = -\mathbf{W}_*^{-1}(t_1)\,\mathbf{z}^\circ \quad (2.6)$$

$$\mathbf{x}(t,\ \mathbf{u}) = \mathbf{X}(t)\left[\mathbf{z}^\circ + \mathbf{W}_*(t)\,\mathbf{u} + \varepsilon \int_0^t \mathbf{X}^{-1}(\sigma)\,\Psi(\mathbf{x}(\sigma,\ \mathbf{u}),\ \sigma)\,d\sigma\right], \qquad 0 \leqslant t \leqslant t_1 \quad (2.7)$$

here $\mathbf{W}_*^{-1}(t_1)$ is the inverse of $\mathbf{W}_*(t_1)$, and $\mathbf{W}_*(t)$ is given by (2.8)

$$\mathbf{W}_*(t) = \|\mathbf{w}_*^\nu(t)\|\,(n \times n), \qquad \mathbf{w}_*^\nu(t) = 1\,(t - \sigma_{\nu-1}) \int_{\sigma_{\nu-1}}^{\sigma_\nu^*(t)} X^{-1}(\sigma)\,\mathbf{b}(\sigma)\,d\sigma \qquad (\nu = 1,\ \ldots,\ n.$$

where

$$\sigma_\nu^*(t) = t + (\sigma_\nu - t)\ 1\,(t - \sigma_\nu)\ (\nu = 1,\ \ldots,\ n) \quad (2.9)$$

$$1\,(t - \sigma) = \begin{cases} 0 & \text{for } t < \sigma \\ 1 & \text{for } t \geqslant \sigma \end{cases} \quad (2.10)$$

Since $t_1 \geqslant \sigma_\nu$ for all ν, we easily see that

$$\sigma_\nu^*(t_1) = \sigma_\nu, \quad 1\,(t_1 - \sigma_{\nu-1}) = 1 \qquad (\nu = 1 \ldots n)$$

Putting in (2, 8) $t = t_1$ we obtain simple expressions for $\mathbf{w}_*^\nu(t_1)$, namely

$$\mathbf{w}_*^{\nu}(t_1) = \int_{\sigma_{\nu-1}}^{\sigma_\nu} \mathbf{X}^{-1}(\sigma)\,\mathbf{b}(\sigma)\,d\sigma \qquad (\nu = 1, \ldots, n;\ \sigma_0 = 0,\ \sigma_n = t_1) \tag{2.11}$$

Although the manner of subdividing the time interval is arbitrary, it must be nonsingular, i. e. the determinant of the matrix $\mathbf{W}_*(t_1)$ must not be equal to zero, otherwise Expression (2. 6) will be meaningless.

Thus, the case considered above, may be reduced by means of nonsingular subdivision of the interval $[0, t_1)$ into n equal subintervals, to the case discussed in Section 1 .

The above method of consecutive approximations can be used to solve (2. 6) and (2. 7). Indeed, let us assume that the k-th approximation is already found and it is such, that

$$|\mathbf{u}^k| \leqslant U^*, \qquad \mathbf{x}(t_1, \mathbf{u}^k) = 0$$

Then, the $(k+1)$-th approximation will be given by

$$\mathbf{u}^{k+1} = \mathbf{u}^\circ - \varepsilon \mathbf{W}_*^{-1}(t_1)\int_0^{t_1} \mathbf{X}^{-1}(t)\,\Psi(\mathbf{x}(t, \mathbf{u}^k), t)\,dt \tag{2.12}$$

$$\mathbf{x}(t, \mathbf{u}^{k+1}) = \mathbf{X}(t)\left[\mathbf{z}^\circ + \mathbf{W}_*(t)\,\mathbf{u}^{k+1} + \varepsilon\int_0^t \mathbf{X}^{-1}(\sigma)\,\Psi(\mathbf{x}(\sigma, \mathbf{u}^k), \sigma)\,d\sigma\right]$$

$$\mathbf{x}(t, \mathbf{u}^\circ) = \mathbf{X}(t)\left[\mathbf{z}^\circ + \mathbf{W}_*(t)\,\mathbf{u}^\circ\right], \qquad 0 \leqslant t \leqslant t_1 \qquad (k = 0, 1, 2, \ldots) \tag{2.13}$$

It is easy to show that if the magnitudes Δ and $\varkappa_\varepsilon$ appearing in the conditions 1° to 3° of Section 1 are, for the above case, given by Formulas

$$\Delta = (1+\delta)\,h^+h^-bU^*t_1,\ \ \delta = \frac{1}{h^-W_*^-bt_1},\ \ b = \max|\mathbf{b}(t)|,\ \ W_*^- = |\mathbf{W}_*^{-1}(t_1)| \qquad t \in [0, t_1]$$

$$\varkappa_\varepsilon = U^* / W_*^- - \varepsilon h^- K t_1 \tag{2.14}$$

while the parameter ε is subject to the condition

$$0 < \varepsilon < \varepsilon_0 = \min\left\{\frac{1}{(1 + h^-W_*^-bt_1)\,h^+h^-rt_1},\ \frac{U^*}{h^-W_*^-Kt_1}\right\} \tag{2.15}$$

then under conditions 1° to 4° with the above alterations incorporated and under the assumption that $\det W_*(t_1) \neq 0$, all approximations (2. 12) for the control vector will satisfy the condition $|\mathbf{u}^{k+1}| \leqslant U^*$, and consequently

$$|u^{k+1}(t)| \leqslant \max|u_\nu^{k+1}| \leqslant |\mathbf{u}^{k+1}| \leqslant U^* \tag{2.16}$$

where

$$u^{k+1}(t) = u_\nu^{k+1}, \qquad \sigma_{\nu-1} \leqslant t \leqslant \sigma_\nu \qquad (\nu = 1, \ldots, n;\ \sigma_0 = 0,\ \sigma_n = t_1)$$

and the sequences of approximations (2. 12) and (2. 13), converge. In other words, under these conditions a control $u^*(t)$ will exist, which will satisfy (2. 2) and will transport the system (2. 1) into the coordinate origin of the phase space, in time t_1 .

It should be noted that a problem of this type on the control of motion of nonlinear systems of more general form, was considered in [3]. Sufficient conditions given there are, however, unconstructive in the sense, that they do not lead to determination of the radius of convergence and the region of controllability. The present paper presents the formulas for determination of the radius of convergence Δ , radius of the sphere of controllability $\varkappa_\varepsilon$ and for the upper bound ε_0 of the values of parameter ε.

3 . As an example, we shall consider the problem on determination of the above quantities for a second order system.

Equations of precessional motion of an Anschütz gyrocompass in presence of nonlinear restoring force, can be represented as [4]

$$dx_1/dt = \mu_2 x_2 + u(t), \quad dx_2/dt = -\mu_1 x_1 - 2\nu x_2 - \varepsilon x_1^3 \tag{3.1}$$

where

$$x_1 = \alpha + \frac{N \sin\varphi}{lP\cos\varphi}, \qquad x_2 = h\left(\beta - \frac{HU\sin\varphi}{lP}\right), \qquad u(t) = -\frac{mU\cos\varphi}{k^2} Q(t)$$

$$\mu_1 = mh\left(\frac{U\cos\varphi}{k}\right)^2, \qquad \mu_2 = \frac{m}{h}, \qquad \nu = \frac{msU\cos\varphi}{k^2}, \qquad 2s = \frac{N}{H} \tag{3.2}$$

Here we employ the notation used in [4] and denote the dimensionless time by t. Using

$$\mathbf{x} = \begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix}, \qquad A = \begin{Vmatrix} 0 & \mu_2 \\ -\mu_1 & -2\nu \end{Vmatrix}, \qquad \Psi(x) = \begin{Vmatrix} 0 \\ -x_1^3 \end{Vmatrix}, \qquad \mathbf{b} = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} \tag{3.3}$$

we shall replace the set of scalar equations (3. 1) with its matrix equivalent

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \varepsilon\Psi(\mathbf{x}) + \mathbf{b}u(t) \tag{3.4}$$

Let us limit ourselves to the case when

$$|u(t)| \leqslant U^*, \qquad 0 \leqslant t \leqslant t_1 \tag{3.5}$$

and consider the problem of determination of Δ, $\varkappa_\epsilon$ and ϵ for which the system (3. 1) is controllable, i. e. for which a control exists satisfying (3. 5) and transporting the system (3. 1) into the position $x = x_2 = 0$ in time t_1.

To solve this problem, we must find the fundamental normalized matrix of (3.1) with $u(t) \equiv 0$ and $\epsilon = 0$. It is easily shown, that this matrix has the form

$$X(t) = e^{-\nu t}\begin{Vmatrix} \cos\omega t + (\nu/\omega)\sin\omega t & (\mu_2/\omega)\sin\omega t \\ -(\mu_1/\omega)\sin\omega t & \cos\omega t - (\nu/\omega)\sin\omega t \end{Vmatrix} \qquad (\omega = \sqrt{\mu_1\mu_2 - \nu^2})$$

Here we have one control and two controlled coordinates, consequently the interval $0 \leq t \leq t_1$ should be divided into two (equal) subintervals.

Using the following numerical values

$$\mu_1 = 0.19, \ \mu_2 = 0.20, \ 2\nu = 0{,}09, \ t_1 = 10, \ U^* = 0.1 \tag{3.7}$$

we can easily show that

$$h^+ = 1{,}41, \ h^- = 2.21, \ W^+ = 0.27, \ \delta = 0.16 \tag{3.8}$$

Computation according to (2. 1) gave $\Delta = 3.61$, and this in turn yielded

$$K = 47.04, \qquad r = 39.09 \tag{3.9}$$

Using the data given in (3. 7) to (3. 9), we obtained $\epsilon_0 = 0.12 \times 10^{-3}$. Taking $\epsilon = 0.11 \times 10^{-3}$ we obtain, by (2. 14), the following value of radius of the sphere of controllability, $\varkappa_\epsilon = 0.29$.

BIBLIOGRAPHY

1. Kalman, R. E., Ob obshchei teorii sistem upravleniia (On the General Theory of Control Systems). Proc. of the 1st IFAK congress, Vol. 2, Izd. Akad. Nauk SSSR, 1961.
2. Roitenberg, Ia. N., Nekotorye zadachi upravleniia dvizheniem (Certain Problems on the Control of Motion). Fizmatgiz, 1963.
3. Rakovshchik, L. S., Postroenie dopustimykh upravlenii I (Construction of admissible controls I). Avtomatika i telemekhanika, Vol. 23, No. 10, 1962.
4. Nguen Tkhan' Bang, K resheniiu odnoi zadachi upravleniia dvizheniem nelineinykh sistem (On the solution of a problem of control of motion of nonlinear systems). Izv. Akad. Nauk SSSR, Tekhnicheskaia kibernetika, No. 5, 1964.

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